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The Inverse Spectral Method for Colliding Gravitational Waves

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Abstract. The problem of colliding gravitational waves gives rise to a Goursat problem in the triangular region $1 \leqslant x < y \leqslant 1$ for a certain 2×2 matrix valued nonlinear equation. This equation, which is a particular exact reduction of the vacuum Einstein equations, is integrable, i.e. it possesses a Lax pair formulation. Using the *simultaneous* spectral analysis of this Lax pair we study the above Goursat problem as well as its linearized version. It is shown that the linear problem reduces to a scalar Riemann–Hilbert problem, which can be solved in closed form, while the nonlinear problem reduces to a 2×2 matrix Riemann–Hilbert problem, which under certain conditions is solvable.

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1. Introduction

One of the most extensively studied problems in general relatively is the collision of two plane gravitational waves in a flat background. Assuming that the two approaching waves are known, it can be shown ([1] and appendix) that the problem of describing the interaction following the collision of the two waves is closely related to the following boundary value problem: Let $g(x, y)$ be a real, symmetric, 2×2 matrix-valued function of x and y for $(x, y) \in D$, where D is the triangular region $D = \{(x, y) \in \mathbb{R}^2, -1 \leqslant x < y \leqslant 1\}$ depicted in Figure 1. Let subscripts denote partial derivatives. The function $g(x, y)$ solves the PDE

$$2(y - x)g_{xy} + g_x - g_y + (x - y)(g_x g^{-1} g_y + g_y g^{-1} g_x) = 0, \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} &g(-1, y) = g_1(y), \quad -1 < y \leqslant 1; \\ &g(x, 1) = g_2(x), \quad -1 \leqslant x < 1, \end{aligned} \tag{1.2}$$

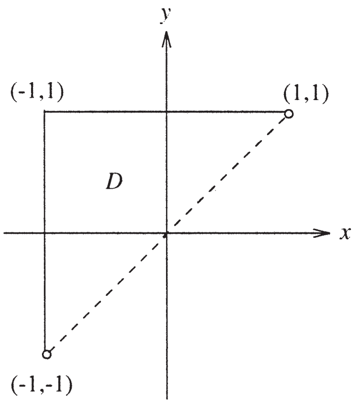

Figure 1. The region $D = \{(x, y) \in \mathbb{R}^2 : -1 \leqslant x < y \leqslant 1\}$ corresponding to the Goursat problem defined by (1.1) and (1.2).

where the functions $g_1(y)$ and $g_2(x)$ are uniquely specified by the approaching waves.

Equation (1.1), which is a particular exact reduction of the vacuum Einstein equations, is equivalent to the celebrated Ernst equation [2]. Belinsky and Zakharov [3] have shown that Equation (1.1) is integrable, in the sense that it admits the Lax pair formulation [4]

$$\frac{\partial \Psi}{\partial x} + \frac{2\kappa}{\kappa + x - y}\frac{\partial \Psi}{\partial \kappa} = \frac{(x-y)g_x g^{-1}}{\kappa + x - y}\Psi, \tag{1.3a}$$

$$\frac{\partial \Psi}{\partial y} + \frac{2\kappa}{\kappa + y - x}\frac{\partial \Psi}{\partial \kappa} = \frac{(y-x)g_y g^{-1}}{\kappa + y - x}\Psi, \tag{1.3b}$$

where $\Psi(x, y, \kappa)$ is a 2×2 matrix-valued function of the arguments indicated and $k \in \mathbb{C}$. An alternative Lax pair of Equation (1.1) is [5, 6]

$$\frac{\partial \Psi}{\partial x} = \frac{1}{2}\left(1 - \frac{(y-\lambda)^{1/2}}{(x-\lambda)^{1/2}}\right) g_x g^{-1}\Psi, \tag{1.4a}$$

$$\frac{\partial \Psi}{\partial y} = \frac{1}{2}\left(1 - \frac{(x-\lambda)^{1/2}}{(y-\lambda)^{1/2}}\right) g_y g^{-1}\Psi, \tag{1.4b}$$

where $\lambda \in \mathbb{C}$. For integrable equations it is usually possible to: (i) Construct a large class of particular explicit solutions, using a variety of the so-called direct methods, such as Bäcklund transformations [7], the dressing method [8], the direct linearizing method [9], etc. (ii) Investigate certain initial-value problems using the so-called inverse spectral method [10–12]. Solving an initial-value problem is more difficult than deriving particular solutions. The problem of colliding gravitational waves is a boundary-value problem and such problems are even more difficult than initial-value problems. Indeed, regarding the interaction of plane gravitational waves, although many classes of particular exact solutions have been found (see [1, 13–17]), the initial-value problem has only been addressed by Hauser and Ernst [18, 19]. These authors did not investigate Equation (1.1) directly. Instead, they have shown that, in the particular case of gravitational waves, Equation (1.1) can be related to the equation

$$(y-x)G_{xy}+[G_x,G_y]=0, \tag{1.5}$$

where [,] denotes the matrix commutator. Equation (1.5) has been studied in [18, 19] using indirectly the fact that Equation (1.5) possesses the Lax pair

$$\frac{\partial\Psi}{\partial x}=\frac{G_x\Psi}{x-\lambda},\qquad \frac{\partial\Psi}{\partial y}=\frac{G_y\Psi}{y-\lambda}, \tag{1.6}$$

where $\Psi(x,y,\lambda)$ is a 2×2 matrix-valued function of the arguments indicated.

In this paper we use the inverse spectral method to solve the boundary value problem defined by Equations (1.1) and (1.2) as well as the following linear boundary value problem: Let the matrix-valued function $\gamma(x,y)$, $(x,y)\in D$, satisfy the linear PDE

$$2(y-x)\gamma_{xy}+\gamma_x-\gamma_y=0, \tag{1.7}$$

where $\gamma(-1,y)$ and $\gamma(x,1)$ are given functions of y and x respectively. This boundary value problem can be considered as the small g limit of the boundary value problem defined by Equations (1.1) and (1.2). Indeed, substituting $g=I+\varepsilon\gamma$ in Equation (1.1) (where I is the identity matrix) and keeping only $O(\varepsilon)$ terms, Equation (1.1) becomes Equation (1.7).

We now state the main result of this paper:

THEOREM 1.1. *Assume that the derivative of* $g_1(y)$ *and of* $g_2(x)$ *are* C^2 *in* $[-1,1]$, *sufficiently small, and* $g_1(1)=g_2(-1)=I$, *where* I *is the* 2×2 *identity matrix. Then the Goursat problem defined by* (1.1) *and* (1.2) *has a unique* C^2 *classical solution in* D. *This solution can be obtained by solving the following Riemann–Hilbert problem for the* 2×2 *matrix-valued functions* Ψ *and* Φ*:*

$$\Psi^+(x,y,\lambda)=\begin{cases}\Phi^-(x,y,\lambda), & x\leqslant\lambda\leqslant y,\\ \Psi^-(x,y,\lambda), & -\infty<\lambda\leqslant -1,\ 1\leqslant\lambda<\infty,\\ \Psi^-(x,y,\lambda)G_l(\lambda), & -1\leqslant\lambda\leqslant x,\\ \Psi^-(x,y,\lambda)G_r(\lambda), & y\leqslant\lambda\leqslant 1,\end{cases} \tag{1.8a}$$

$$\Phi^+(x,y,\lambda)=\begin{cases}\Psi^-(x,y,\lambda), & x\leqslant\lambda\leqslant y,\\ \Phi^-(x,y,\lambda), & -\infty<\lambda\leqslant -1,\ 1\leqslant\lambda<\infty,\\ \Phi^-(x,y,\lambda)G_l(\lambda)^{-1}, & -1\leqslant\lambda\leqslant x,\\ \Phi^-(x,y,\lambda)G_r(\lambda)^{-1}, & y\leqslant\lambda\leqslant 1,\end{cases} \tag{1.8b}$$

$$\lim_{\lambda\to\infty}\Psi=I,\qquad \lim_{\lambda\to\infty}\Phi=g, \tag{1.8c}$$

where

$$\Psi^{\pm}(x,y,\lambda)=\Psi(x,y,\lambda\pm i0),\qquad \Phi^{\pm}(x,y,\lambda)=\Phi(x,y,\lambda\pm i0),\quad \lambda\in\mathbb{R},$$

and the 2×2 *matrix-valued functions* $G_l(\lambda)$, $G_r(\lambda)$ *are defined in terms of* $g_1(y)$ *and* $g_2(x)$ *as follows:*

$$G_l(\lambda) = (L_-(\lambda, \lambda))^{-1} L_+(\lambda, \lambda), \qquad G_r(\lambda) = (R_-(\lambda, \lambda))^{-1} R_+(\lambda, \lambda), \tag{1.9}$$

where

$$\begin{aligned} L_\pm(x, \lambda) &= I + \frac{1}{2} \int_{-1}^{x} \left(1 \mp i \frac{(1-\lambda)^{1/2}}{(\lambda - \xi)^{1/2}} \right) \times \\ &\quad \times \frac{\mathrm{d}g_2(\xi)}{\mathrm{d}\xi} g_2^{-1}(\xi) L_\pm(\xi, \lambda)\, \mathrm{d}\xi, \quad -1 \leqslant x \leqslant \lambda, \end{aligned} \tag{1.10}$$

$$\begin{aligned} R_\pm(y, \lambda) &= I - \frac{1}{2} \int_{y}^{1} \left(1 \pm i \frac{(\lambda+1)^{1/2}}{(\eta - \lambda)^{1/2}} \right) \times \\ &\quad \times \frac{\mathrm{d}g_1(\eta)}{\mathrm{d}\eta} g_1^{-1}(\eta) R_\pm(\eta, \lambda)\, \mathrm{d}\eta, \quad \lambda \leqslant y \leqslant 1. \end{aligned} \tag{1.11}$$

We conclude this introduction with some remarks.

(1) It can be shown that if $g \in \mathbb{R}$, and if $g(x, y)$ satisfies the equation $gCgC = \rho(x-y)^2$, where ρ is a real constant and C is a real, nonsingular, constant matrix, then Equation (1.1) is simply related to (1.5). This relationship is valid for the particular case of gravity. Thus the initial-value problem for the colliding gravitational waves can also be investigated by applying the inverse spectral method to Equations (1.6). The inverse spectral method for the Lax pair (1.6) involves the technical difficulty of analyzing eigenfunctions with Cauchy type singularities; a rigorous investigation of this problem remains open.

(2) The boundary value problem of Equation (1.7) mentioned above was first solved by Szekeres [20] using the classical Riemann function technique. The same problem was later solved by Hauser and Ernst using separation of variables and the Abel transform [21, 22]. The linear problem has also been discussed in [23]. It was emphasized in [24] that before solving a given nonlinear integrable equation, it is quite useful to use the inverse spectral method to solve the linearized version of this nonlinear equation. This is carried out in Section 3, using the fact that Equation (1.7) possesses the Lax pair

$$\frac{\partial \Psi}{\partial x} + \frac{\Psi}{2(x-\lambda)} = \frac{\gamma_x}{2(x-\lambda)}, \qquad \frac{\partial \Psi}{\partial y} + \frac{\Psi}{2(y-\lambda)} = \frac{\gamma_y}{2(y-\lambda)}. \tag{1.12}$$

(3) When analyzing a Lax pair, it is customary to study the two equations forming this pair *independently*. Indeed, one usually studies one of the two equations to formulate an inverse problem in terms of appropriate spectral data, and then one uses the second equation to determine the "evolution" of the spectral data. Actually, this philosophy is precisely the one used for solving linear equations. However, it turns out that for solving the boundary value problem (1.1) and (1.2), it is more convenient to study both equations forming the Lax pair *simultaneously*. This important insight was gained from the inverse spectral analysis of the linear

Equation (1.7). The simultaneous spectral analysis of the Lax pair has led to a unified transform method for solving initial-boundary-value problems for linear and for integrable nonlinear PDE's [25].

(4) The Riemann–Hilbert (RH) problem (1.8) has the technical difficulty that $\Phi(x, y, \infty) = g(x, y)$ is unknown. This difficulty can be bypassed by formulating an *equivalent* RH problem for some other sectionally analytic functional μ (see Equation (3.16) for the relationship between Φ, Ψ and μ). The function μ satisfies $\mu(x, y, \infty) = I$. Furthermore, it is shown in [27] that the RH problem satisfied by μ is solvable *without* a small norm assumption of $g_1(y)$ and of $g_2(x)$ provided that they satisfy a certain symmetry condition. This is a consequence of the fact that there exists a *topological vanishing lemma* for this RH problem (see [27] for details). We emphasize that the solvability of the RH problem (1.8) is based on the proof presented in [27] of the solvability of the equivalent RH problem for μ.

2. The Lax Pair Representation

PROPOSITION 2.1. *Let $g(x, y)$ be a matrix-valued function belonging to $C^2(\mathbb{R}^2)$.*

- (i) *The nonlinear Equation* (1.1) *is the compatibility condition of Equations* (1.3)*, where $\Psi(x, y, \kappa)$ is a 2×2 matrix-valued function belonging to $C^2(\mathbb{R} \times \mathbb{C})$ and $\kappa \in \mathbb{C}$.*
- (ii) *Equation* (1.1) *is also the compatibility condition of Equations* (1.4)*.*
- (iii) *Under the transformation*

$$G_x = (x - y)g_x g^{-1}, \qquad G_y = (y - x)g_y g^{-1}, \tag{2.1}$$

Equation (1.1) *becomes*

$$2(y - x)G_{xy} + G_y - G_x + [G_x, G_y] = 0. \tag{2.2}$$

Proof. (i) and (iii). Let Ψ satisfy

$$\Psi_x + \frac{2\kappa}{\kappa + x - y}\Psi_\kappa = \frac{A\Psi}{\kappa + x - y}, \tag{2.3a}$$

$$\Psi_y + \frac{2\kappa}{\kappa + y - x}\Psi_\kappa = \frac{B\Psi}{\kappa + y - x}, \tag{2.3b}$$

where $A(x, y)$ and $B(x, y) \in C^1(\mathbb{R}^2)$. It can be verified that the compatibility of Equations (2.3) yields

$$A_y = B_x, \tag{2.4a}$$

$$(x - y)(B_x + A_y) + A - B + [B, A] = 0. \tag{2.4b}$$

Indeed, if

$$D_1 := \partial_x + \frac{2\kappa}{\kappa + x - y}\partial_\kappa, \qquad D_2 := \partial_y + \frac{2\kappa}{\kappa + y - x}\partial_\kappa,$$

it is straightforward to show that

$$D_1(D_2\Psi) = D_2(D_1\Psi). \tag{2.5}$$

Then the compatibility of Equations (2.3) yields

$$D_2\left(\frac{A\Psi}{\kappa + x - y}\right) = D_1\left(\frac{B\Psi}{\kappa + y - x}\right),$$

which implies Equations (2.4).

Integrating Equation (2.4a) it follows that $A = G_x$ and $B = G_y$. Then Equation (2.4b) becomes Equation (2.2). Using Equations (2.1) in Equation (2.2), Equation (1.1) follows. We note that the compatibility condition of Equations (2.1) is Equation (1.1) itself, thus the transformation (2.1) is well-defined.

(ii) It can be verified directly that the compatibility of Equations (1.4) is Equation (1.1). □

REMARK 2.1. Equation (2.2) is the compatibility condition of

$$\Psi_x = \frac{1}{2}\left(1 - \left(\frac{y-\lambda}{x-\lambda}\right)^{1/2}\right)G_x\Psi, \tag{2.6a}$$

and

$$\Psi_y = \frac{1}{2}\left(1 - \left(\frac{x-\lambda}{y-\lambda}\right)^{1/2}\right)G_y\Psi. \tag{2.6b}$$

REMARK 2.2. It is straightforward to obtain the Lax pair (1.4) from the Lax pair (1.3): Indeed, one can introduce characteristic coordinates in (1.3) if and only if

$$\frac{\partial\kappa}{\partial x} = \frac{2\kappa}{\kappa + x - y}, \qquad \frac{\partial\kappa}{\partial y} = \frac{2\kappa}{\kappa + y - x}. \tag{2.7}$$

These equations are compatible since $\kappa_{xy} = 2\kappa/(\kappa^2 - (x-y)^2) = \kappa_{yx}$. Their solution is

$$\kappa^2 + 2\kappa(2\lambda - (x+y)) + (x-y)^2 = 0,$$

where λ is a constant. Thus

$$\kappa = x + y - \lambda + 2(x-\lambda)^{1/2}(y-\lambda)^{1/2}. \tag{2.8}$$

Using this equation it follows that

$$\frac{\kappa + x - y}{2} = (x-\lambda)^{1/2}\big((x-\lambda)^{1/2} + (y-\lambda)^{1/2}\big),$$

and

$$\frac{\kappa + y - x}{2} = (y-\lambda)^{1/2}\big((x-\lambda)^{1/2} + (y-\lambda)^{1/2}\big).$$

Substituting these expressions into the right-hand side of Equations (1.3), and using

$$\begin{aligned} x - y &= (x-\lambda) - (y-\lambda) \\ &= \big((x-\lambda)^{1/2} - (y-\lambda)^{1/2}\big)\big((x-\lambda)^{1/2} + (y-\lambda)^{1/2}\big), \end{aligned}$$

Equations (1.3) become Equations (1.4).

PROPOSITION 2.2. *Let $g(x, y)$ satisfy Equation* (1.1). *Assume that*

$$g \in \mathbb{R}, \qquad gCgC = \rho(x-y)^2, \tag{2.9}$$

where ρ is a real constant, and C is a real nonsingular constant matrix. Define $G(x, y)$ by

$$G = i\alpha gC + \beta f C, \quad \alpha^2 = -\frac{1}{16\rho},\ \beta = -\frac{1}{4\rho\nu},\ \nu = \text{constant}, \tag{2.10}$$

where $f(x, y)$ is defined by

$$f_x = \frac{\nu gCg_x}{x-y}, \qquad f_y = \frac{\nu gCg_y}{y-x}. \tag{2.11}$$

Then $G(x, y)$ solves Equation (1.5).

Proof. Equation (2.9) implies

$$g_xCgC + gCg_xC = 2\rho(x-y), \tag{2.12}$$
$$g_yCgC + gCg_yC = 2\rho(y-x). \tag{2.13}$$

Using $g^{-1} = CgC/\rho(x-y)^2$, Equation (1.1) becomes

$$(x-y)\left(2g_{xy} - \frac{g_xCgCg_y}{\rho(x-y)^2} - \frac{g_yCgCg_x}{\rho(x-y)^2}\right) + g_y - g_x = 0.$$

Multiplying this equation by gC and using Equations (2.12) to replace gCg_xC and gCg_yC, it follows that

$$(x-y)(2gCg_{xy} + g_yCg_x + g_xCg_y) + gCg_x - gCg_y = 0. \tag{2.14}$$

This equation can be written as

$$\left(\frac{gCg_x}{x-y}\right)_y = \left(\frac{gCg_y}{y-x}\right)_x,$$

which shows that f is well-defined by Equation (2.11).

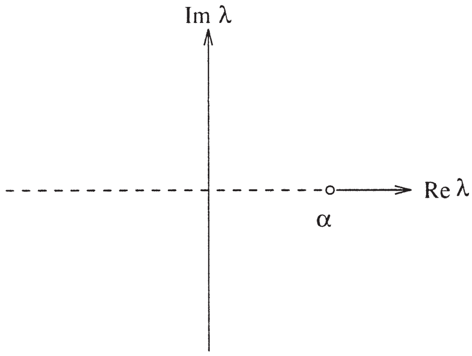

Figure 2. The cut complex λ-plane used in Theorem 3.1.

Substituting $G = i\alpha gC + \beta fC$ into (1.5) one finds two equations. One of them is

$$(y-x)g_{xy} + \beta g_x Cf_y - \beta g_y Cf_x + \beta f_x Cg_y - \beta f_y Cg_x = 0.$$

Replacing f_x and f_y (see Equations (2.11)), gCg_xC and gCg_yC (see Equations (2.12)), and CgC by $g^{-1}\rho(x-y)^2$, this equation becomes (1.1) if and only if $4\rho\beta\nu = -1$. The other equation is

$$\beta(y-x)f_{xy} - \alpha^2 g_x Cg_y + \alpha^2 g_y Cg_x + \beta^2 f_x Cf_y - \beta^2 f_y Cf_x = 0.$$

Replacing f, gCg_xC and gCg_yC, this equation becomes (2.13) if and only if $4\alpha^2 = \nu\beta$. □

3. The Spectral Theory of a Boundary Value Problem of the Ernst Equation

We first discuss the linear equation (1.7).

THEOREM 3.1. *Let the matrix-valued function $\gamma(x, y)$, where $-1 \leqslant x < y \leqslant 1$, satisfy Equation* (1.7)*. Let $\gamma(-1, y)$ and $\gamma(x, 1)$ be given differentiable functions of y and x respectively. The solution of this boundary value problem is given by*

$$\begin{aligned}\gamma(x, y) \;=\; & \gamma(-1, 1) + \frac{1}{\pi}\int_{-1}^{x} \frac{(1-\lambda)^{1/2}}{(x-\lambda)^{1/2}(y-\lambda)^{1/2}}\hat{\gamma}_1(\lambda)\,\mathrm{d}\lambda - \\ & -\frac{1}{\pi}\int_{y}^{1} \frac{(\lambda+1)^{1/2}}{(\lambda-x)^{1/2}(\lambda-y)^{1/2}}\hat{\gamma}_2(\lambda)\,\mathrm{d}\lambda,\end{aligned} \tag{3.1}$$

where

$$\hat{\gamma}_1(\lambda) = \int_{-1}^{\lambda} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\gamma(x, 1)}{(\lambda - x)^{1/2}}\,\mathrm{d}x, \qquad \hat{\gamma}_2(\lambda) = \int_{\lambda}^{1} \frac{\frac{\mathrm{d}}{\mathrm{d}y}\gamma(-1, y)}{(y-\lambda)^{1/2}}\,\mathrm{d}y. \tag{3.2}$$

Proof. Let the function $(\lambda-\alpha)^{1/2}$, $\alpha\in\mathbb{R}$, be defined with respect to a branch cut from $-\infty$ to α (see Figure 2).

The common solution of Equations (1.12) satisfying $\Psi(-1,1,\lambda)=0$, possesses two different representations,

$$\Psi=\begin{cases}\dfrac{1}{(\lambda-x)^{1/2}}\displaystyle\int_{-1}^{x}\frac{\gamma_{x'}\,\mathrm{d}x'}{2(\lambda-x')^{1/2}}-\\ \quad-\dfrac{(\lambda+1)^{1/2}}{(\lambda-x)^{1/2}(\lambda-y)^{1/2}}\displaystyle\int_{y}^{1}\frac{\gamma_{y'}(-1,y')}{2(\lambda-y')^{1/2}}\,\mathrm{d}y', & (3.3a)\\ -\dfrac{1}{(\lambda-y)^{1/2}}\displaystyle\int_{y}^{1}\frac{\gamma_{y'}\,\mathrm{d}y'}{2(\lambda-y')^{1/2}}+\\ \quad+\dfrac{(\lambda-1)^{1/2}}{(\lambda-x)^{1/2}(\lambda-y)^{1/2}}\displaystyle\int_{-1}^{x}\frac{\gamma_{x'}(x',1)}{2(\lambda-x')^{1/2}}\,\mathrm{d}x'. & (3.3b)\end{cases}$$

Using

$$-1\leqslant x'\leqslant x<y\leqslant y'\leqslant 1,$$

it follows that:

(i) If $\lambda>1$, the square roots appearing in (3.3) have no jumps, hence Ψ has no jumps.

(ii) If $\lambda<-1$, all the square roots have jumps, which however cancel, and hence Ψ has no jumps.

(iii) If $-1\leqslant\lambda\leqslant x$, then $(\lambda-y)^{1/2}$, $(\lambda-y')^{1/2}$, $(\lambda-1)^{1/2}$, $(\lambda-x)^{1/2}$ have jumps, $(\lambda+1)^{1/2}$ has no jump, and $(\lambda-x')^{1/2}$ has no jump if $\lambda>x'$ but has a jump if $\lambda<x'$. Thus, if the superscripts $+$ and $-$ denote the limit of Ψ as λ approaches the real axis from above and below respectively, Equation (3.3b) implies

$$\Psi^+-\Psi^-=\frac{(1-\lambda)^{1/2}}{i(x-\lambda)^{1/2}(y-\lambda)^{1/2}}\int_{-1}^{\lambda}\frac{\gamma_{x'}(x',1)}{(\lambda-x')^{1/2}}\,\mathrm{d}x',\quad -1\leqslant\lambda\leqslant x. \tag{3.4}$$

(iv) Similarly if $y\leqslant\lambda\leqslant 1$, Equation (3.3a) yields

$$\Psi^+-\Psi^-=-\frac{(\lambda+1)^{1/2}}{i(\lambda-x)^{1/2}(\lambda-y)^{1/2}}\int_{\lambda}^{1}\frac{\gamma_{y'}(-1,y')}{(y'-\lambda)^{1/2}}\,\mathrm{d}y',\quad y\leqslant\lambda\leqslant 1. \tag{3.5}$$

Thus Ψ is a sectionally holomorphic function of λ, with jumps only in $[-1,x]$ and $[y,1]$, given by Equations (3.4) and (3.5) respectively. Also Equations (3.3) imply that $\Psi=\mathrm{O}(\frac{1}{\lambda})$ as $\lambda\to\infty$, $\lambda_I\neq 0$. This information defines a Riemann–Hilbert problem [26] for Ψ. Its unique solution is given by

$$\Psi=-\frac{1}{2\pi}\int_{-1}^{x}\frac{(1-\lambda')^{1/2}}{(x-\lambda')^{1/2}(y-\lambda')^{1/2}}\hat{\gamma}_1(\lambda')\frac{\mathrm{d}\lambda'}{\lambda'-\lambda}+$$

$$+\frac{1}{2\pi}\int_y^1 \frac{(\lambda'+1)^{1/2}}{(\lambda'-x)^{1/2}(\lambda'-y)^{1/2}}\hat{\gamma}_2(\lambda')\frac{d\lambda'}{\lambda'-\lambda}, \quad \lambda_I \neq 0. \tag{3.6}$$

Equation (3.3) implies that

$$\Psi = \frac{1}{2\lambda}\big[\gamma(x,y)-\gamma(-1,1)\big] + \mathrm{O}\left(\frac{1}{\lambda^2}\right), \quad \lambda_I \neq 0,\ \lambda \to \infty. \tag{3.7}$$

Using Equation (3.6) to compute the $\mathrm{O}(\frac{1}{\lambda})$ term of Ψ and comparing with Equation (3.7), Equation (3.1) follows.

The rigorous justification of the above formalism involves the following steps:

(i) Given $\gamma(-1,y)$ and $\gamma(x,1)$ in C^1, Equations (3.2) define $\hat{\gamma}_1(\lambda)$ and $\hat{\gamma}_2(\lambda)$ in C^1.

(ii) Given $\hat{\gamma}_1(\lambda)$ and $\hat{\gamma}_2(\lambda)$ in C^1, define $\gamma(x,y)$ by Equation (3.1). Use a direct computation to show that $\gamma(x,y)$ satisfies Equation (1.7) and the given boundary conditions. □

Derivation of Theorem 1.1. We first *assume* that $g(x,y)$ exists and show that $g(x,y)$ can be obtained through the solution of the RH problem (1.8). We then discuss the rigorous justification of this construction *without* the a priori assumption of existence.

Let $\Psi(x,y,\lambda)$ be the unique matrix-valued function defined by

$$\Psi_x = \frac{1}{2}\left(1-\frac{(\lambda-y)^{1/2}}{(\lambda-x)^{1/2}}\right)g_x g^{-1}\Psi, \tag{3.8a}$$

$$\Psi_y = \frac{1}{2}\left(1-\frac{(\lambda-x)^{1/2}}{(\lambda-y)^{1/2}}\right)g_y g^{-1}\Psi, \tag{3.8b}$$

$$\Psi(-1,1,\lambda) = I. \tag{3.8c}$$

Ψ possesses the two different integral representations

$$\Psi = I + \begin{cases} \displaystyle\int_{-1}^x \left(1-\frac{(\lambda-y)^{1/2}}{(\lambda-x')^{1/2}}\right)a(x',y)\Psi(x',y,\lambda)\,dx' - \\ \displaystyle\quad -\int_y^1 \left(1-\frac{(\lambda+1)^{1/2}}{(\lambda-y')^{1/2}}\right)b(-1,y')\Psi(-1,y',\lambda)\,dy', \\ \displaystyle -\int_y^1 \left(1-\frac{(\lambda-x)^{1/2}}{(\lambda-y')^{1/2}}\right)b(x,y')\Psi(x,y',\lambda)\,dy' + \\ \displaystyle\quad +\int_{-1}^x \left(1-\frac{(\lambda-1)^{1/2}}{(\lambda-x')^{1/2}}\right)a(x',1)\Psi(x',1,\lambda)\,dx', \end{cases} \tag{3.9}$$

where $a(x,y)$ and $b(x,y)$ are defined by

$$a(x,y) = \frac{1}{2}g_x g^{-1}, \qquad b(x,y) = \frac{1}{2}g_y g^{-1}. \tag{3.10}$$

Note that $\Psi(x, 1, \lambda)$ and $\Psi(-1, y, \lambda)$ satisfy

$$\Psi(x, 1, \lambda) = I + \frac{1}{2}\int_{-1}^{x}\left(1 - \frac{(\lambda-1)^{1/2}}{(\lambda-x')^{1/2}}\right) \times \\ \times (g_x g^{-1})(x', 1)\Psi(x', 1, \lambda)\,\mathrm{d}x' \tag{3.11}$$

and

$$\Psi(-1, y, \lambda) = I - \frac{1}{2}\int_{y}^{1}\left(1 - \frac{(\lambda+1)^{1/2}}{(\lambda-y')^{1/2}}\right) \times \\ \times (g_y g^{-1})(-1, y')\Psi(-1, y', \lambda)\,\mathrm{d}y'. \tag{3.12}$$

Let $\Phi(x, y, \lambda)$ be the unique matrix-valued function defined by

$$\Phi_x = \frac{1}{2}\left(1 + \frac{(\lambda-y)^{1/2}}{(\lambda-x)^{1/2}}\right) g_x g^{-1}\Phi, \tag{3.13a}$$

$$\Phi_y = \frac{1}{2}\left(1 + \frac{(\lambda-x)^{1/2}}{(\lambda-y)^{1/2}}\right) g_y g^{-1}\Phi, \tag{3.13b}$$

$$\Phi(-1, 1, \lambda) = I. \tag{3.13c}$$

$\Phi(x, y, \lambda)$, $\Phi(x, 1, \lambda)$ and $\Phi(-1, y, \lambda)$ satisfy equations analogous to Equations (3.9), (3.11) and (3.12).

We now compute the jumps of Ψ.

(i) $\lambda < -1$ or $\lambda > 1$.
The integral representations (3.9) imply that $\Psi^+ = \Psi^-$. Indeed, if $\lambda > 1$ none of the square roots appearing in (3.9) has a jump; if $\lambda < -1$ all of the square roots have a jump, which however cancel.

(ii) $-1 \leqslant \lambda \leqslant x$.
Both $(\lambda - y)^{1/2}$ and $(\lambda - x)^{1/2}$ have a jump, hence $(\lambda - y)^{1/2}/(\lambda - x)^{1/2}$ has no jump and both Ψ^+ and Ψ^- satisfy Equations (3.8). Thus

$$\Psi^+(x, y, \lambda) = \Psi^-(x, y, \lambda)G_l(\lambda).$$

In order to compute the matrix $G_l(\lambda)$ we evaluate this equation at $x = \lambda$ and $y = 1$,

$$G_l(\lambda) = (\Psi^-(\lambda, 1, \lambda))^{-1}\Psi^+(\lambda, 1, \lambda).$$

Let $L_\pm(s, \lambda) = \lim_{z\to\lambda\pm i0}\Psi(s, 1, z)$ for $-1 \leqslant s \leqslant \lambda$. Equation (3.11) yields

$$L_\pm(s, \lambda) = I + \frac{1}{2}\int_{-1}^{s}\left(1 \mp i\frac{(1-\lambda)^{1/2}}{(\lambda-s')^{1/2}}\right) \times \\ \times (g_x g^{-1})(s', 1)L_\pm(s', \lambda)\,\mathrm{d}s' \tag{3.14}$$

for $-1 \leqslant s \leqslant \lambda$ and $\Psi^\pm(\lambda, 1, \lambda) = L_\pm(\lambda, \lambda)$. We have thus established (1.10) and the first half of (1.9).

(iii) $y \leqslant \lambda \leqslant 1$.
Both $(\lambda - y)^{1/2}$ and $(\lambda - x)^{1/2}$ have no jumps, hence $(\lambda - y)^{1/2}/(\lambda - x)^{1/2}$ has no jump and both Ψ^+ and Ψ^- satisfy Equations (3.8). Thus

$$\Psi^+(x, y, \lambda) = \Psi^-(x, y, \lambda)G_r(\lambda).$$

In order to compute the matrix $G_r(\lambda)$ we evaluate this equation at $y = \lambda$ and $x = -1$,

$$G_r(\lambda) = (\Psi^-(-1, \lambda, \lambda))^{-1}\Psi^+(-1, \lambda, \lambda).$$

Let $R_\pm(t, \lambda) = \lim_{z\to\lambda\pm i0} \Psi(-1, t, z)$ for $\lambda \leqslant t \leqslant 1$. Equation (3.12) yields

$$R^\pm(t, \lambda) = I - \frac{1}{2}\int_t^1 \left(1 \pm i\frac{(\lambda + 1)^{1/2}}{(t' - \lambda)^{1/2}}\right) \times \times (g_y g^{-1})(-1, t')R_\pm(t', \lambda)\,\mathrm{d}t' \tag{3.15}$$

for $\lambda \leqslant t \leqslant 1$ and $\Psi^\pm(-1, \lambda, \lambda) = R_\pm(\lambda, \lambda)$. We have thus established (1.11) and the second half of (1.9).

(iv) $x \leqslant \lambda \leqslant y$.
The ratio $(\lambda - y)^{1/2}/(\lambda - x)^{1/2}$ has a jump, thus Ψ^+ and Φ^- satisfy the same system of integrable equations. Since $\Psi^+(-1, 1, \lambda) = I = \Phi^-(-1, 1, \lambda)$, we have $\Psi^+ = \Phi^-$.

The jumps of Φ can be computed in a similar way. Also note that for $-1 \leqslant x \leqslant \lambda$ or $\lambda \leqslant y \leqslant 1$.

$$\Psi^\pm(x, 1, \lambda) = \Phi^\mp(x, 1, \lambda), \qquad \Psi^\pm(-1, y, \lambda) = \Phi^\mp(-1, y, \lambda).$$

Equations (3.9) and the analogous equation for Φ imply Equation (1.8c).

We now discuss the rigorous justification of the above construction:

(i) Equations (1.10) and (1.11) are Volterra integral equations. Thus if g_1 and $g_2 \in C^2$, the jump matrices G_l and G_r are well-defined.

(ii) It can be shown that the RH problem (1.8) has a unique global solution. This follows from the fact that this RH problem is simply related to a RH problem satisfied by the function $\mu(x, y, w)$ defined by

$$\mu(x, y, w) = \begin{cases} \Phi(x, y, f(x, y, w)), & \left|w - \frac{1}{2}\right| \leqslant \frac{1}{2}, \\ \Psi(x, y, f(x, y, w)), & \left|w - \frac{1}{2}\right| \geqslant \frac{1}{2}, \end{cases} \tag{3.16}$$

where $\lambda = f(x, y, w)$ is the rational function defined by

$$\left(1 - \frac{1}{w}\right)^2 = \frac{\lambda - y}{\lambda - x}. \tag{3.17}$$

The function μ satisfies $\mu(x, y, \infty) = I$, furthermore it turns out that the RH problem for μ satisfies a vanishing lemma [27], i.e., the homogeneous RH problem has only the zero solution, provided that g and g_2 satisfy a certain symmetry condition.

(iii) Using direct differentiation it can be shown that if Φ and Ψ solve the RH problem (1.8), then

$$\Psi_x = \frac{1}{2}\left[1 - \frac{(\lambda - y)^{1/2}}{(\lambda - x)^{1/2}}\right]\alpha(x, y)\Psi, \quad \Psi_y = \frac{1}{2}\left[1 - \frac{(\lambda - x)^{1/2}}{(\lambda - y)^{1/2}}\right]\beta(x, y)\Psi, \tag{3.18}$$

$$\Phi_x = \frac{1}{2}\left[1 + \frac{(\lambda - y)^{1/2}}{(\lambda - x)^{1/2}}\right]\alpha(x, y)\Phi, \quad \Phi_y = \frac{1}{2}\left[1 + \frac{(\lambda - x)^{1/2}}{(\lambda - y)^{1/2}}\right]\beta(x, y)\Phi, \tag{3.19}$$

where α and β are some λ-independent functions. Let Ψ_1 and g be defined by

$$\Psi(x, y, \lambda) = I + \frac{\Psi_1(x, y)}{\lambda} + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty, \quad \Phi(x, y, \lambda) = g(x, y) + o(1) \quad \text{as } \lambda \to \infty. \tag{3.20}$$

Then

$$(\Psi_1)_x = \frac{(y - x)}{4}\alpha(x, y), \qquad (\Psi_2)_x = \frac{(x - y)}{4}\beta(x, y), \tag{3.21}$$

$$\alpha(x, y) = g_x g^{-1}, \qquad \beta(x, y) = g_y g^{-1}. \tag{3.22}$$

The compatibility condition of Equation (3.21) implies that g solves the Ernst equation.

(iv) The proof that $g(x, y)$ satisfies $g(-1, y) = g_1(y)$ and $g(x, 1) = g_2(x)$ is given in [27].

(v) Equation (1.1) is invariant under

$$g \to gA, \qquad g \to \bar{g}, \qquad g \to g^T, \qquad g \to g^{-1},$$

where A is a nonsingular matrix. Thus without loss of generality we can assume that $g(-1, 1) = I$. Furthermore, if $g_1(y)$ and $g_2(x)$ are real, symmetric, positive definite matrices, then the solution also has the same properties. □

Appendix. The Collision of Two Plane Gravitational Waves

The spacetime manifold representing the collision of plane gravitational waves in vacuum is characterized by the presence of two spacelike, commuting and hypersurface orthogonal Killing vector fields. This allows one to write the metric as

$$ds^2 = g_{ab}\, dx^a\, dx^b - 2f\, du\, dv, \quad a, b = 1, 2, \tag{A.1}$$

where the 2×2 symmetric matrix function $g := (g_{ab})$ and the scalar function f depend only on the null coordinates u, v, and satisfy the constraints

$$f(u, v) > 0, \qquad \det(g(u, v)) > 0. \tag{A.2}$$

Hence, one can introduce a pair of scalar functions α and Γ such that

$$\det g = \alpha^2, \quad \alpha(u, v) > 0 \quad \text{and} \quad f = \alpha^{-1/2} \mathrm{e}^{2\Gamma}. \tag{A.3}$$

Thus, the matrix g can be written as

$$g = \alpha S, \quad \det S = 1, \tag{A.4}$$

and Equation (A.1) takes the form

$$\mathrm{d}s^2 = \alpha S_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b - 2\alpha^{-1/2} \mathrm{e}^{2\Gamma}\, \mathrm{d}u\, \mathrm{d}v. \tag{A.5}$$

In this form the four degrees of freedom characterizing the geometry of the space-time manifold of plane gravitational waves are expressed by the two scalar functions α and Γ and the unimodular, symmetric 2×2 matrix S. The two degrees of freedom incorporated in the latter can be expressed by a pair of real valued functions F and ω. Thus, S can be written as

$$S = F^{-1} \begin{bmatrix} E\bar{E} & \omega \\ \omega & 1 \end{bmatrix}, \quad E := F + i\omega, \tag{A.6}$$

where $\bar{E}$ denotes the complex conjugate of E.

The functions α, Γ and E are determined by solving the Einstein field equations in the vacuum, namely the system $R_{ij}(u, v) = 0,\ i, j = 1, 2, 3, 4$, where R_{ij} is the Ricci tensor corresponding to Equation (A.5). The components of this system which do not vanish identically yield

$$\alpha_{u,v} = 0, \tag{A.7}$$

$$F(2\alpha E_{uv} + \alpha_u E_v + \alpha_v E_u) = 2\alpha E_u E_v, \tag{A.8}$$

$$\Gamma_u = \frac{1}{2}\frac{\alpha_{uu}}{\alpha_u} + \frac{\alpha}{\alpha_u}\left|\frac{E_u}{2F}\right|^2, \tag{A.9a}$$

$$\Gamma_v = \frac{1}{2}\frac{\alpha_{vv}}{\alpha_v} + \frac{\alpha}{\alpha_v}\left|\frac{E_v}{2F}\right|^2, \tag{A.9b}$$

$$\Gamma_{uv} = -\mathrm{Re}\left(\frac{E_u \bar{E}_v}{4F^2}\right). \tag{A.10}$$

The fundamental components of the above system of field equations are Equations (A.7) and (A.8). This follows from the fact that Equations (A.7) and (A.8)

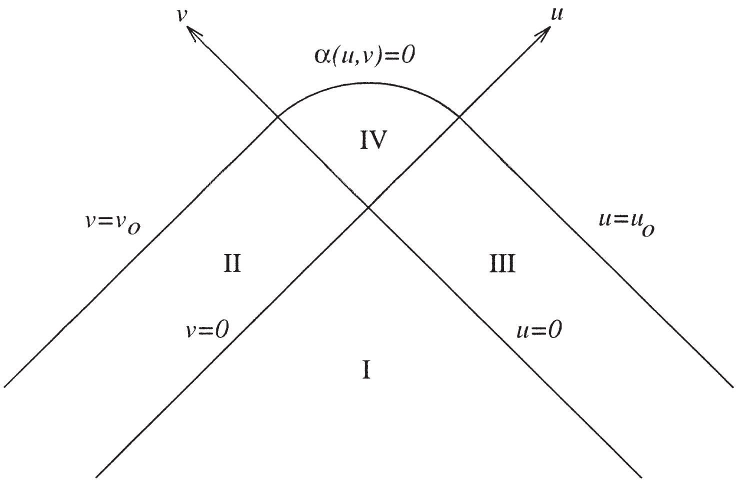

Figure 3. The domain $W = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$ of the (u, v)-plane corresponding to a space-time manifold which represents the collision of a pair of plane gravitational waves. Region I represents the initially flat (gravity free) domain into which the waves propagate. The two incoming pulses of gravitational radiation are represented by region II and III, respectively. Their interaction is represented by region IV, which corresponds to region D of Figure 1.

are the integrability conditions of Equations (A.9) and (A.10). Hence, given α and E, Γ can be found by quadrature.

The matrix equation

$$(\alpha g_u g^{-1})_v + \alpha(g_v g^{-1})_u = 0, \tag{A.11}$$

called the *Ernst equation*, is equivalent to the system of Equations (A.7) and (A.8). In particular, taking the trace of Equation (A.11) one finds Equation (A.7).

Let us now consider the following adjacent regions of the (u, v)-plane (see Figure 3), where (u_0, v_0) is a pair of positive numbers,

$$\begin{aligned}
\mathrm{I} &= \left\{(u, v) \in \mathbb{R}^2 : u \leqslant 0, v \leqslant 0\right\},\\
\mathrm{II} &= \left\{(u, v) \in \mathbb{R}^2 : u \leqslant 0, 0 \leqslant v < v_0\right\},\\
\mathrm{III} &= \left\{(u, v) \in \mathbb{R}^2 : 0 \leqslant u < u_0, v \leqslant 0\right\},\\
\mathrm{IV} &= \left\{(u, v) \in \mathbb{R}^2 : 0 \leqslant u < u_0, 0 \leqslant v < v_0, \alpha(u, v) > 0\right\}.
\end{aligned}$$

It will be assumed that the metric coefficients are continuous in the domain $W := \mathrm{I} \cup \mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$, with $\alpha(u, v) > 0$ for all $(u, v) \in W$, and $\alpha(u, v) = 0$ for $(u, v) \in \partial W$. Moreover, the same symbols I–IV will be used in the following for the corresponding regions of space-time. For example, II denotes the set $\mathrm{II} \times \mathbb{R}^2$, where $\mathbb{R}^2$ represents the extent of the ignorable coordinates x^1 and x^2.

Region I represents a domain free of gravity into which a pair of gravitational waves impinge from the left and from the right. The latter are represented by regions II and III, respectively. Thus, in region I the line element is given by

$$\mathrm{d}s_{\mathrm{I}}^2 = -2\,\mathrm{d}u\,\mathrm{d}v + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2. \tag{A.12}$$

In region II the metric coefficients depend only on v. They are specified by a given u-independent solution of the field equations (A.7)–(A.10). Similarly, the metric coefficients in region III depend only on u, and follow from a given v-independent solution of the same equations. By continuity, the given solutions in regions II and III determine the initial values of the metric coefficients in region IV, i.e., their values along the null hypersurfaces $u = 0$, $0 \leqslant v < v_0$ and $0 \leqslant u < u_0$, $v = 0$. Thus, taking into account the earlier remarks regarding the function Γ, one can formulate the problem associated with the process of colliding plane gravitational waves as follows. Find $(\alpha(u, v), E(u, v))$ which: (i) satisfy Equations (A.7) and (A.8) in the interior of region IV, and (ii) take preassigned values along the boundary ∂IV of the above region, where $\partial\mathrm{IV} = \{(u, v) \in \mathbb{R}^2\colon u = 0,\ 0 \leqslant v < v_0\} \cup \{(u, v) \in \mathbb{R}^2\colon 0 \leqslant u < u_0, v = 0\}$. It is assumed that the boundary data sets $\{\alpha(0, v), \alpha(u, 0)\}$ and $\{E(0, v), E(u, 0)\}$ consist of functions which belong to the differentiability classes C^2 and C^1, respectively.

Following [21], let us introduce the functions r, s defined by

$$r(u) := 1 - 2\alpha(u, 0), \quad 0 \leqslant u < u_0, \tag{A.13a}$$

$$s(v) := 2\alpha(0, v) - 1, \quad 0 \leqslant v < v_0. \tag{A.13b}$$

Then it is easily verified that the unique solution of Equation (A.7) in region IV which satisfies the given initial conditions is given by

$$\alpha(u, v) = \frac{1}{2}\big(s(v) - r(u)\big). \tag{A.14}$$

It turns out that the field equations themselves determine a set of junction conditions along the null hypersurfaces $u = 0$ and $v = 0$. Following [21] these conditions can be written in the following form

(i)

$$\frac{\mathrm{d}r}{\mathrm{d}u}(u) > 0, \quad \text{for } 0 < u < u_0, \tag{A.15a}$$

$$\frac{\mathrm{d}s}{\mathrm{d}v}(v) < 0, \quad \text{for } 0 < v < v_0, \tag{A.15b}$$

(ii)

$$\frac{\mathrm{d}r}{\mathrm{d}u}(0) = \frac{\mathrm{d}s}{\mathrm{d}v}(0) = 0. \tag{A.16}$$

(iii) The following limits exist

$$\lim_{u\to 0^+}\left[\frac{\frac{\mathrm{d}^2 r}{\mathrm{d}u^2}(u) - 4L(u, 0)}{2\frac{\mathrm{d}r}{\mathrm{d}u}(u)}\right], \quad \text{where } L := \alpha\left|\frac{E_u}{2F}\right|^2, \tag{A.17a}$$

$$\lim_{v\to 0^+}\left[\frac{\frac{\mathrm{d}^2 s}{\mathrm{d}v^2}(v) - 4K(0, v)}{2\frac{\mathrm{d}s}{\mathrm{d}v}(v)}\right], \quad \text{where } K := \alpha\left|\frac{E_v}{2F}\right|^2. \tag{A.17b}$$

Conditions (ii) and (iii), called *colliding wave conditions* by Hauser and Ernst, must be satisfied in order for a solution of the associated boundary-value problem to admit the interpretation of a colliding plane gravitational wave model. Condition (i), on the other hand, allows one to introduce a new pair of null coordinates x, y by setting

$$x = r(u), \qquad y = s(v). \tag{A.18}$$

These equations define a one-to-one, bicontinuous mapping of region IV of the (u, v)-plane onto the triangular region $D = \{(x, y) \in \mathbb{R}^2\colon -1 \leqslant x < y \leqslant 1\}$ of the (x, y)-plane.

In the new coordinate system $\alpha = \frac{1}{2}(y - x)$, and Equation (A.11) becomes

$$\left(\frac{1}{2}(y - x)g_x g^{-1}\right)_y + \left(\frac{1}{2}(y - x)g_y g^{-1}\right)_x = 0. \tag{A.19}$$

Thus, the boundary-value problem reduces to solving Equation (A.19), which is equivalent to Equation (1.1), in the interior of D for specified boundary data $E(-1, y)$ and $E(x, 1)$.

Global aspects of this problem and the singularity structure of the corresponding space-time manifolds are discussed in [28, 29].

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